MATRIX AND DETERMINANTS DIVISION USING SALIHU'S METHOD

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ABSTRACT

In this paper we will present division of matrices and determinants. At division of determinants the determinants must have the same order $n\ x\ n$ and the result will again be determinant of the same order $n \times n$, and the final result is the same if we calculate determinants. While the division of the matrix as a result we will have a matrix of the same order $m \times n$ with the first respectively second matrix depending on whether it is right or left division. In order to apply this formula for division of matrices, they must have the same number of columns on right division or same number of rows on left division, the divisor matrix should be square matrix and its determinant should be different from zero.

Keywords: *Division of matrices, Division of determinants, Left division, Right Division.*

1. INTRODUCTION

1.1. Matrices

Matrix is a table with real or complex numbers listed in m rows and n columns. Symbolically written as:

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

Number a_{ij} $(i = 1, 2, \dots, m, j = 1, 2, \dots, n)$ are elements of the matrix, index i and j are used to identify rows/columns in which is listed the element.

1.2 Matrix multiplication

Let $[A_{mxn}]$ and $[B_{nxn}]$ be two matrices of order $m\ x\ n$, respectively $n\ x\ n$. Multiplication of matrices $[A_{mxn}]$ and $[B_{nxn}]$ is matrix $[C_{mxn}]$ described below:

$$[C_{mxn}] = \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ c_{m1} & c_{m2} & \cdots & c_{mn} \end{bmatrix}$$

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{i1} & a_{i2} & \dots & a_{in} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \cdot \begin{bmatrix} b_{11} & b_{12} & \dots & b_{1j} & \dots & b_{1n} \\ b_{21} & b_{22} & \dots & b_{2j} & \dots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \dots & b_{nj} & \dots & b_{nn} \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1j} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2j} & \cdots & c_{2n} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ c_{i1} & c_{i2} & \cdots & c_{ij} & \cdots & c_{in} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ c_{m1} & c_{m2} & \cdots & c_{mj} & \cdots & c_{mn} \end{bmatrix}$$

$$c_{ij} = a_{i1} \cdot b_{1j} + a_{i2} \cdot b_{2j} + \cdots + a_{in} . b_{nj} = \sum_{k=1}^{n} a_{ik} \cdot b_{kj}$$

Second matrix is a square matrix that means the matrix has the same order of rows and columns.

1.3. Inverse matrix

Let $[A_{nxn}]$ be a square matrix of order $n\ x\ n$, than the square matrix $[B_{nxn}]$ is the inverse matrix of $[A_{nxn}]$ if is meet the condition below:

$[A_{nxn}]\cdot[B_{nxn}] = [B_{nxn}]\cdot[A_{nxn}] = E$, where E is unit matrix

If $[A_{nxn}]^{-1}$ exist, than $[A_{nxn}]^{-1}$ is unique matrix.

1.4. Minors of matrices

If $[A_{nxn}]$ is a square matrix, than minor of element in row i and column j is the determinant formed when removing row i and column j, while signe + or – is determined from $(-1)^{i+j}$:

$$[A] = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Than minor M_{23} is:

$$M_{23} = (-1)^{2+3}\cdot Det\begin{bmatrix} a_{11} & a_{12} & x \\ x & x & x \\ a_{31} & a_{32} & x \end{bmatrix} = (-1)^{2+3}\cdot Det\begin{bmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{bmatrix}$$

1.5. Determinants

Let A be a $n\ x\ n$ matrix:

$$[A_{nxn}] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

Than determinant of order $n\ x\ n$ is the sum:

$$|A_{nxn}| = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = \sum_{S_n} \varepsilon_{j_1 j_2 \dots j_n}\cdot a_{j_1}\cdot a_{j_2}\cdot \dots \cdot a_{j_n}$$

Ranging over the symmetric permutation group S_n, where:

$$\varepsilon_{j_1 j_2 \dots j_n} = \begin{cases} +1, \text{if } j_1 j_2 \dots j_n, \text{is an even permutation} \\ -1, \text{if } j_1 j_2 \dots j_n, \text{is an odd permutation} \end{cases}$$

2. DETERMINANTS DIVISION

Let $|A_{nxn}|$ and $|B_{nxn}|$ be two determinants of order $n\ x\ n$:

$$|A_{nxn}| = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} \text{ and } |B_{nxn}| = \begin{vmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nn} \end{vmatrix}$$

Division of determinants $|A_{nxn}|$ and $|B_{nxn}|$ gives the determinant $|C_{nxn}|$, respectively:

$$|A_{nxn}|/|B_{nxn}| = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} \Bigg/ \begin{vmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nn} \end{vmatrix} = \begin{vmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ c_{n1} & c_{n2} & \cdots & c_{nn} \end{vmatrix} = |C_{nxn}|$$

Where division is done for every element according to the formula below:

$$\begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{i1} & a_{i2} & \dots & a_{in} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix} \Bigg/ \begin{vmatrix} b_{11} & b_{12} & \dots & b_{1n} \\ b_{21} & b_{22} & \dots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{j1} & b_{j2} & \dots & b_{jn} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \dots & b_{nn} \end{vmatrix} = \begin{vmatrix} c_{11} & c_{12} & \cdots & c_{1j} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2j} & \cdots & c_{2n} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ c_{i1} & c_{i2} & \cdots & c_{ij} & \cdots & c_{in} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ c_{n1} & c_{n2} & \cdots & c_{nj} & \cdots & c_{nn} \end{vmatrix}$$

$$c_{ij} = \frac{a_{i1} \cdot M(b_{j1}) + a_{i2} \cdot M(b_{j2}) + \dots + a_{in} \cdot M(b_{jn})}{|B_{nxn}|} = \sum_{k=1}^{n} \frac{a_{ik} \cdot M(b_{jk})}{|B_{nxn}|}$$

In order to be used this formula for determinants division, both determinants must have the same order and the second/divisor determinant must be different from zero, $|B_{nxn}| \neq 0$

Example 1.
Are given determinants $|A|$ and $|B|$ of third order:

$$|A| = \begin{vmatrix} 3 & 5 & -2 \\ -4 & 1 & 3 \\ -1 & 2 & 2 \end{vmatrix} = 27 \text{ and } |B| = \begin{vmatrix} 1 & 0 & -1 \\ 0 & 4 & 1 \\ 3 & 3 & 0 \end{vmatrix} = 9$$

Division of determinants $|A|$ and $|B|$ is:

$$C = |A|/|B| = \begin{vmatrix} 3 & 5 & -2 \\ -4 & 1 & 3 \\ -1 & 2 & 2 \end{vmatrix} \Bigg/ \begin{vmatrix} 1 & 0 & -1 \\ 0 & 4 & 1 \\ 3 & 3 & 0 \end{vmatrix} = \frac{27}{9} = 3$$

But there are some times that we need as a result to have again determinant, than we can use method described above:

$$|C| = |A|/|B| = \begin{vmatrix} 3 & 5 & -2 \\ -4 & 1 & 3 \\ -1 & 2 & 2 \end{vmatrix} \Bigg/ \begin{vmatrix} 1 & 0 & -1 \\ 0 & 4 & 1 \\ 3 & 3 & 0 \end{vmatrix} = \begin{vmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{vmatrix}$$

$$|C| = \begin{vmatrix} \frac{30}{9} & \frac{12}{9} & -\frac{1}{9} \\ -\frac{21}{9} & \frac{6}{9} & -\frac{5}{9} \\ -\frac{15}{9} & \frac{3}{9} & \frac{2}{9} \end{vmatrix} = 3$$

3. MATRIX DIVISION

3.1. Right division of matrices

Let $[A_{mxn}]$ be a matrix of order $m\ x\ n$ and $[B_{nxn}]$ a square matrix of order $m\ x\ m$.

Right division of matrices can be done according to the formula below:

$$[A_{mxn}]/[B_{mxm}] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \Bigg/ \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nn} \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ c_{m1} & c_{m2} & \cdots & c_{mn} \end{bmatrix}$$

Where division is done for every element as described below:

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \Big/ \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{j1} & b_{j2} & \cdots & b_{jn} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nn} \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1j} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2j} & \cdots & c_{2n} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ c_{i1} & c_{i2} & \cdots & c_{ij} & \cdots & c_{in} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ c_{m1} & c_{m2} & \cdots & c_{mj} & \cdots & c_{mn} \end{bmatrix}$$

$$c_{ij} = \frac{a_{i1} \cdot M(b_{j1}) + a_{i2} \cdot M(b_{j2}) + \cdots + a_{in} \cdot M(b_{jn})}{|B_{nxn}|} = \sum_{k=1}^{n} \frac{a_{ik} \cdot M(b_{jk})}{|B_{nxn}|}$$

In order to apply this formula for the right division of matrices, both matrices must have the same number of columns and second/divisor matrix should be a square matrix and its determinants have to be different from zero, $|B_{nxn}| \neq 0$ Result is the same if compared with other used forms for solving those mathematical problems.

Example 2: Are given matrices $[A_{3x4}]$ and $[B_{3x3}]$:

$$[A] = \begin{bmatrix} 2 & 5 & 1 \\ -3 & 1 & 0 \\ 2 & -3 & 4 \\ 7 & -2 & -5 \end{bmatrix} \; and \; [B] = \begin{bmatrix} -4 & 5 & 3 \\ 8 & -1 & -2 \\ -3 & 1 & 1 \end{bmatrix}$$

Right division of matrices can be found:

$$[A]/[B] = \begin{bmatrix} 2 & 5 & 1 \\ -3 & 1 & 0 \\ 2 & -3 & 4 \\ 7 & -2 & -5 \end{bmatrix} \Big/ \begin{bmatrix} -4 & 5 & 3 \\ 8 & -1 & -2 \\ -3 & 1 & 1 \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \\ c_{41} & c_{42} & c_{43} \end{bmatrix}$$

$$[C] = \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \\ c_{41} & c_{42} & c_{43} \end{bmatrix} = \begin{bmatrix} -3 & 10 & 30 \\ -5 & 11 & 37 \\ 28 & -63 & -206 \\ -14 & 31 & 99 \end{bmatrix}$$

By using above formula the whole process is division and we don't need to solve inverse matrix.

The result also can be proven by multiplying matrices with inverse function:

$$[C] \cdot [B] = [A]$$

$$\begin{bmatrix} -3 & 10 & 30 \\ -5 & 11 & 37 \\ 28 & -63 & -206 \\ -14 & 31 & 99 \end{bmatrix} \cdot \begin{bmatrix} -4 & 5 & 3 \\ 8 & -1 & -2 \\ -3 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 5 & 1 \\ -3 & 1 & 0 \\ 2 & -3 & 4 \\ 7 & -2 & -5 \end{bmatrix}$$

3.2. Left division of matrices

Let $[A_{nxn}]$ be a square matrix of order $n \, x \, n$ and $[B_{nxm}]$ be a matrix of order $n \, x \, m$.

Left division of matrices can be done according to the formula below:

$$[A_{nxn}] \backslash [B_{nxm}] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \Big\backslash \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1m} \\ b_{21} & b_{22} & \cdots & b_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nm} \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1m} \\ c_{21} & c_{22} & \cdots & c_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ c_{n1} & c_{n2} & \cdots & c_{nm} \end{bmatrix}$$

Where division is done for every element as described below:

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1i} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2i} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{ni} & \cdots & a_{nn} \end{bmatrix} \Bigg\backslash \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1j} & \cdots & b_{1m} \\ b_{21} & b_{22} & \cdots & b_{2j} & \cdots & b_{2m} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nj} & \cdots & b_{nm} \end{bmatrix}$$

$$= \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1j} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2j} & \cdots & c_{2n} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ c_{i1} & c_{i2} & \cdots & \boxed{c_{ij}} & \cdots & c_{in} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ c_{m1} & c_{m2} & \cdots & c_{mj} & \cdots & c_{mn} \end{bmatrix}$$

$$c_{ij} = \frac{b_{1j} \cdot M(a_{1i}) + b_{2j} \cdot M(a_{2i}) + \cdots + b_{nj} \cdot M(a_{ni})}{|A_{nxn}|} = \sum_{k=1}^{n} \frac{b_{kj} \cdot M(a_{ki})}{|A_{nxn}|}$$

In order to apply this formula for the left division of matrices, both matrices must have the same number of rows and firs/divisor matrix should be a square matrix and its determinants have to be different from zero, $|A_{nxn}| \neq 0$
Result is the same if compared with other used forms for solving those mathematical problems.

Example 3: Are given matrices $[A_{3x3}]$ and $[B_{4x3}]$:

$$[A] = \begin{bmatrix} -4 & 5 & 3 \\ 8 & -1 & -2 \\ -3 & 1 & 1 \end{bmatrix} \; and \; [B] = \begin{bmatrix} 2 & -3 & 2 & 7 \\ 5 & 1 & -3 & -2 \\ 1 & 0 & 4 & -5 \end{bmatrix}$$

Right division of matrices can be found:

$$[A]\backslash[B] = \begin{bmatrix} -4 & 5 & 3 \\ 8 & -1 & -2 \\ -3 & 1 & 1 \end{bmatrix} \Bigg\backslash \begin{bmatrix} 2 & -3 & 2 & 7 \\ 5 & 1 & -3 & -2 \\ 1 & 0 & 4 & -5 \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & c_{13} & c_{14} \\ c_{21} & c_{22} & c_{23} & c_{24} \\ c_{31} & c_{32} & c_{33} & c_{34} \end{bmatrix}$$

$$[C] = \begin{bmatrix} c_{11} & c_{12} & c_{13} & c_{14} \\ c_{21} & c_{22} & c_{23} & c_{24} \\ c_{31} & c_{32} & c_{33} & c_{34} \end{bmatrix} = \begin{bmatrix} -15 & -5 & -20 & 46 \\ 37 & 11 & 45 & -104 \\ -81 & -26 & -101 & 237 \end{bmatrix}$$

By using above formula the whole process is division and we don’t need to solve inverse matrix.

The result also can be proven by multiplying matrices with inverse function:

$$[A] \cdot [C] = [B]$$

$$\begin{bmatrix} -4 & 5 & 3 \\ 8 & -1 & -2 \\ -3 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} -15 & -5 & -20 & 46 \\ 37 & 11 & 45 & -104 \\ -81 & -26 & -101 & 237 \end{bmatrix} = \begin{bmatrix} 2 & -3 & 2 & 7 \\ 5 & 1 & -3 & -2 \\ 1 & 0 & 4 & -5 \end{bmatrix}$$

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