Global Weight Domination on Duplicated S-Valued Graphs

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Abstract— Now a days, the application domination in graphs lies in various fields in solving real life problems. It includes social network theory, surveying, radio stations, school bus routing, computer communication networks etc. So it has attracted many researchers to work on it. Many variants of dominating sets are available in existing literature. In [7] the authors introduced the notion of semiring-valued graphs. They also studied weight dominating vertex set and weight domination number on S-valued graphs. Motivated by this, in this paper we investigate global set domination on S- valued graphs.

Keywords— Dominating set, complement of a S-valued graph, global domination set, duplication of vertex, $\gamma^S - \mathbf{set}$, $\gamma_g^S - \mathbf{set}$

I. INTRODUCTION

The Domination problem was studied from the 1950s onwards, but the rate of research on domination significantly increased in the mid-1970s. Domination sets are of practical interest in several areas. So it has attracted many researchers to work on it. Many variants of dominating sets are available in existing literature. The concept of global domination in a graph was introduced by Sampathkumar [9]. Duplication of a vertex v of a graph G produces a new graph G' by adding a vertex v' with N(v') = N(v). In other words a vertex v' is said to be duplication of v if all the vertices which are adjacent to v are now adjacent to v' also. If the vertices of a graph G are duplicated altogether then the resultant graph is known as splitting graph of G which is denoted as S'(G).

In [7] the authors introduced the notion of semiring-valued graphs(S-valued graphs) and studied weight dominating vertex set, global weight domination vertex set, weight domination and global weight domination number on S-valued graphs[4,5]. A subset D⊆V of a S-valued graph is called a weight dominating vertex set of G^s if for each v∈ D $\sigma(u) \preccurlyeq \sigma(v)$ ∀ u∈N^s[v]. The minimum cardinality of a weight dominating set of G^s is called a weight domination number of G^s which is denoted by $\gamma^S(G^S)$ and the corresponding weight dominating set is called a $\gamma^S -$ set of G^s. A weight dominating set D⊆V of a S-valued graph G^S is said to be a global weight dominating set if D is also a weight dominating set in the complement of $\overline{G^S}$. The minimum cardinality of a global weight dominating set of G^s is called a global weight domination number of G^s which is denoted by $\gamma_g^S(G^S)$ and the corresponding global weight dominating set is called a $\gamma_g^S -$ set of G^s.

In the present work we investigate some general results which relate the concepts of global weight domination, duplication of a vertex and complement of S-valued graphs.

II. PRELIMINARIES

Definition 2.1. [6] A Semiring (S,+,.) is an algebraic system with a non empty set S together with + and • such that
1.(S,+,0) is a monoid.
2.(S,•) is a semigroup.
3.For all a,b,c ∈ S, a•(b+c)=a•b+a•c and (a+b)•c=a•c+b•c
4.0•x=x•0=0∀ x∈ S

Definition 2.2. [7] Let (S,+,•) be a semiring. A relation ≼ is said to be a canonical pre-order if for a,b ∈ S, a≼b if and only if there exists c∈S such that a+c =b

Definition 2.3. [7] Let G =(V,E⊂ VXV) be the underlying graph with both V,E≠ ∅. For any semiring (S,+,•) a semiring valued graph (or an S-valued graph) G^s is defined to be the graph G^s=(V,E,σ,ψ) where σ:V→S and ψ:E→S is defined to be

$$\psi(x,y) = \begin{cases} \min(\sigma(x),\sigma(y)), & if\ \sigma(x) \preccurlyeq \sigma(y)\ or\ \sigma(y) \preccurlyeq \sigma(x) \\ 0 & otherwise \end{cases}$$

For every unordered pair (x,y) of E⊂ VXV we call σ a S-vertex set and ψan S-edge set of S-valued graph G^s

Definition 2.4. [4] Consider the S-valued graph G^s=(V,E,σ,ψ). For v_i ∈V,the open neighbourhood of v_i in G^s is defined as a subset of VxS such that $N^s[v_i]$={(v_j, $\sigma(v_j)$/(v_i,v_j) ∈E, $\psi(v_i, v_{j)})$ ∈S}.

For $v_i \in V$ an closed neighbourhood of v_i in G^s is defined to be the subset of VXS such that $N^s[v_j]$= $N^s[v_j]$∪{ v_j, $\sigma(v_i)$}

Definition 2.5. [4] Let G^s=(V,E,σ,ψ) be a given S-valued graph. A vertex v in G^s is said to be a weight dominating vertex if $\sigma(u) \preccurlyeq \sigma(v)$ $\forall u$ ∈N^s[v]

Definition 2.6. [4] A subset D⊆V is called a weight dominating vertex set of G^s if for each v∈ D $\sigma(u) \preccurlyeq \sigma(v)$ ∀ u∈N^s[v].

The minimum cardinality of a weight dominating set of G^s is called a weight domination number of G^s which is denoted by $\gamma^S(G^S)$ and the corresponding weight dominating set is called a $\gamma^S -$ set of G^s.

Definition 2.7. [1] A closed trail whose origin and internal vertices are distinct is called a cycle.

For S-valued cycle G^s, C_n^s denotes a cycle on n vertices

Definition 2.8. [1] The complement $\bar{G}$ of a simple graph G is the simple graph with vertex set V, two vertices being adjacent in $\bar{G}$ iff they are not adjacent in G.

Definition 3.2. [5] A weight dominating set D⊆V of a S-valued graph G^S is said to be a global weight dominating set if D is also a weight dominating set in the complement of $\overline{G^S}$.

The minimum cardinality of a global weight dominating set of G^s is called a global weight domination number of G^s which is denoted by $\gamma_g^S(G^S)$ and the corresponding global weight dominating set is called a $\gamma_g^S -$ set of G^s.

Definition 2.9. [4] A Dominating set D⊆V of a graph G is said to be a global dominating set if D is also a dominating set in the complement of G.

Definition 2.10. [8] Duplication of a vertex v of a graph G produces a new graph G' by adding a vertex v' with N(v')= N(v). In other words a vertex v' is said to be duplication of v if all the vertices which are adjacent to v are now adjacent to v' also.

If the vertices of a graph G are duplicated altogether then the resultant graph is known as splitting graph of G which is denoted by $S'(G)$.

III. DUPLICATION OF A VERTEX IN AN S-VALUED GRAPH

Definition 3.1. Duplication of a vertex (v, σ(v)) of a S- valued graph G^S is an S-valued graph obtained by adding a vertex (v',σ(v')) such that $N_s(v')$= N_s(v).

In other words a vertex (v', σ(v')) is said to be the duplication of (v, σ(v)) if all the vertices which are adjacent to v in G^s are also adjacent to v' in $(G^S)'$.

Before moving to the results, let us see some examples.

Example 3.3. Let (S={0,a,b,c},+, .} be a semiring with the following cayley tables.

+	0	a	b	c
0	0	a	b	c
a	a	a	b	c
b	b	b	b	b
c	c	c	b	c

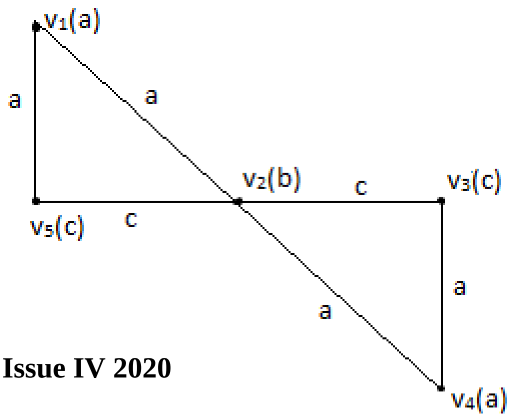
•	0	a	b	c
0	0	0	0	0
a	0	0	a	0
b	0	a	b	c
c	0	0	c	c

Here ≼ is a canonical pre order in S given by 0≼0, 0≼a, 0≼b, 0≼c,a≼a,a≼b, a≼c,b≼b, c≼c,c≼b

Consider the S-valued graph G^s=(V,E,σ,Ψ) where σ:V→S and ψ:E→S are defined to be

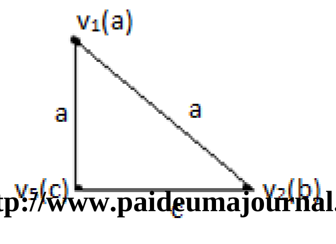
σ(v_1)=σ(v_4)=a:σ(v_2)=b and σ(v_3)=σ(v_5)=c

Ψ(v_1,v_2)=Ψ(v_1,v_5)=Ψ(v_2,v_4)=a and Ψ(v_2,v_3)=Ψ(v_2,v_5)=c



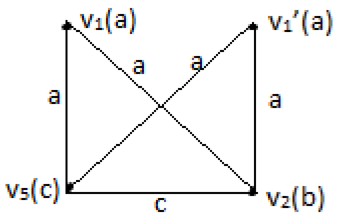
G^s

Let us consider a S-valued cycle on 3 vertices C_3^s



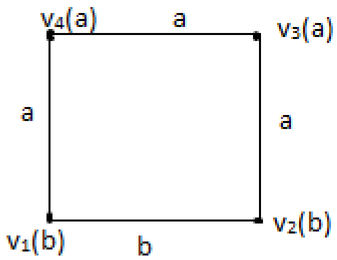
Now, $N_s[v_2]=\{(v_1,a),(v_5,c),(v_2,b)\}$ Also, $\sigma(v_1)=a\preccurlyeq b=\sigma(v_2)$, $\sigma(v_5)=c\preccurlyeq b=\sigma(v_2)$.
So, v_2 is a weight dominating vertex in G^S.
We construct a new graph of C_3^s by duplicating the vertex v_1.

We get the following duplicated graph $(C_3^s)'$



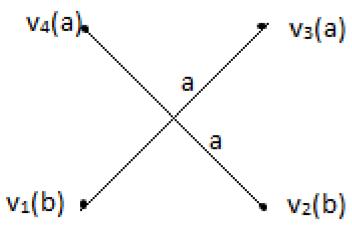
In this duplication graph $\psi(v_1',v_5)=\min\{\sigma(v_1'),\sigma(v_5)\}=a$
$\psi(v_1',v_2)=\min\{\sigma(v_1'),\sigma(v_2)\}=a$
$N_s[v_2]=\{(v_1',a),(v_1,a),(v_5,c),(v_2,b)\}$
$\sigma(v_1')=a\preccurlyeq b=\sigma(v_2)$
$\sigma(v_1)=a\preccurlyeq b=\sigma(v_2)$
$\sigma(v_5)=c\preccurlyeq b=\sigma(v_2)$
Clearly v_2 is also a weight dominating vertex for this duplication graph.
Hence, if we take a weight dominating vertex in a cycle C_3^s of a S-valued graph G^S then it is also a weight dominating vertex in its duplication graph $(C_3^s)'$.

Example 3.4.
Suppose we take a S-cycle C_4^s with the same semiring as example 3.3



Here, $N_s[v_1]=\{(v_4,a),(v_1,b),(v_2,b)\}$ such that $\sigma(v_4)=a\preccurlyeq b=\sigma(v_1)$ and $\sigma(v_2)=b\preccurlyeq b=\sigma(v_1)$.
Also, $N_s[v_2]=\{(v_1,b),(v_3,a),(v_2,b)\}$ such that $\sigma(v_1)=b\preccurlyeq b=\sigma(v_2)$ and $\sigma(v_3)=a\preccurlyeq b=\sigma(v_2)$
Hence, $D=\{v_1,v_2\}$ is a weight dominating vertex set of the cycle C_4^s.

We take the complement of C_4^s



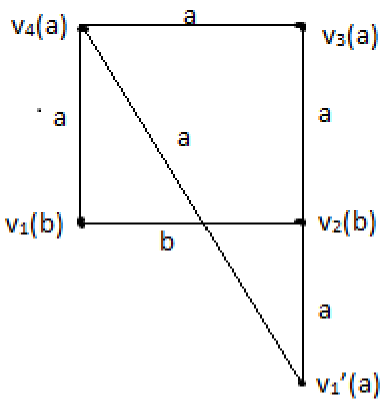
$\overline{C_4^S}$

Here, $N_s[v_1]=\{(v_1,b),(v_3,a)\}$ such that $\sigma(v_3)=a\preccurlyeq b=\sigma(v_1)$
and $N_s[v_2]=\{(v_4,a),(v_2,b)\}$ such that $\sigma(v_4)=a\preccurlyeq b=\sigma(v_2)$
So $D=\{v_1,v_2\}$ is a weight dominating vertex set of $\overline{C_4^S}$.

Hence D={v_1,v_2} is a global weight dominating vertex set of C_4^s.

Example3.5.

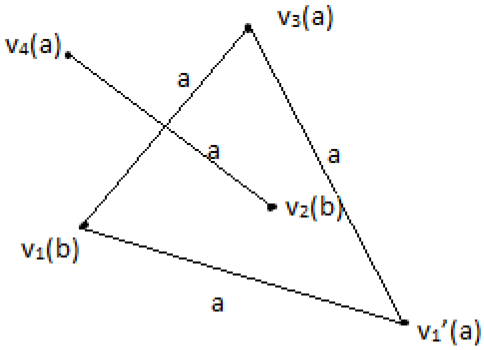
If we construct a duplication graph of the above graph $C_4^s$by duplicating the vertex v_1 we get the following graph $(C_4^s)'$



$(C_4^s)'$

In this graph, $N_s[v_1]$={(v_1,b),(v_4,a),(v_2,b)}, σ(v_4)=a≼b=σ(v_1) and σ(v_2)=b≼b=σ(v_1)
Also, $N_s[v_2]$={(v_1,b),(v_2,b),(v_3,a),(v'$_1$,a)}, σ(v_1)=b≼b=σ(v_2) and σ(v_3)=a≼b=σ(v_2)
Thus D={v_1,v_2}is a weight dominating vertex set of duplicate graph also.

Complement of this duplicate graph is



$\overline{(C_4^s)'}$

In this complemented duplicate graph, $N_s[v_1]$={(v_1,b),(v_3,a),(v'$_1$,a)}. σ(v_3)=a≼b=σ(v_1) and σ(v_1')=a≼b=σ(v_1).
Also, $N_s[v_2]$={(v_4,a),(v_2,b)}, σ(v_4)=a≼b=σ(v_2)
So {v_1,v_2} is a weight dominating set of the complement of this duplicate graph $\overline{(C_4^s)'}$.

IV. PROPERTIES OF DUPLICATION OF A VERTEX IN AN S-VALUED GRAPH

Theorem 4.1

For any S-valued cycle of p vertices C_p^s, let $(C_p^s)'$ be the graph obtained by duplication of a vertex v by v' where $v \in V(C_p^s)$. If D is a weight dominating vertex set of C_p^s containing either of the vertices which are adjacent to v then D is also a weight dominating vertex set of $(C_p^s)'$.

Proof:

If $v \in V(C_p^s)$ is duplicated by v' then V($C_p^{s'}$) = V(C_p^s) ∪{ v',σ(v')}
If D is a weight dominating vertex set of C_p^s and if we take (x, σ(x)) ∈ D which dominates $v \in V$ then x is adjacent to v' in $(C_p^s)'$
Thus D is also a dominating set of $(C_p^s)'$.

Theorem 4.2

Let $(C_p^s)'$ be the graph obtained by the duplication of a vertex v of C_p^s by v'.If D is a global dominating vertex set of C_p^s containing either of the vertices which are adjacent to v then D is also a global weight dominating vertex set of $(C_p^s)'$.

Proof:

Let D be a global weight dominating vertex set of C_p^s

Then D is also a weight dominating vertex set of C_p^s (By definition)

Also D is also a weight dominating vertex set of $(C_p^s)'$ (By Theorem 4.1)

So it is enough to prove that D is a global weight dominating vertex set of $(C_p^s)'$

That is it is enough to prove that D is a weight dominating vertex set of $\overline{(C_p^s)'}$

Take a vertex u∈D which is adjacent to v in $(C_p^s)'$. Then it is not adjacent to v in $\overline{(C_p^s)'}$

Now, as D is a weight dominating vertex set of $\overline{C_p^s}$, there exists a vertex x∈D and u≠x such that x is not adjacent to v in C_p^s but dominates v in $\overline{C_p^s}$.

Since u∈D is adjacent to both v and v' in $(C_p^s)'$, it is not adjacent to both v and v' in $\overline{(C_p^s)'}$.

Also, V($\overline{(C_p^s)'}$)=V($\overline{C_p^s}$)∪ $\{v'\}$ and D is a weight dominating vertex set of $\overline{C_p^s}$, the vertex x∈D must dominate v' in $\overline{(C_p^s)'}$

Hence D is a weight dominating vertex set of $\overline{(C_p^s)'}$.

Theorem 4.3

If D is a γ^S- set of P_n^S(n≥4) then D is a global dominating set of P_n^S. Also $\gamma^S(P_n^S)= \gamma_g^S(P_n^S)$.

Proof

For P_n^S (n≥4) consider a γ^S- set $D=\begin{cases}\{v_2, v_5, \ldots\ldots\ldots v_{3j+2}\} & \text{if } n \equiv 0 \text{ or } 2 (\text{mod } 3)\\ \{v_2, v_5, \ldots\ldots\ldots\ldots v_{3j+2}\} \cup \{v_{n-1}\} & if\ n \equiv 1\ mod\ (3)\end{cases}$

where $0 \le j \le \lfloor\frac{n-2}{3}\rfloor$

Here we label the vertices of G^s in such a way that the pendant vertices have the same labelling say $b \in S$ and all the non pendant vertices have same value a $\in S$ such that b≼a.

In P_n^S there are two vertices of degree 1 and (n-2) internal vertices are of degree 2.

Now let $(v_i, \sigma(v_i))\epsilon D$ and $(v_j, \sigma(v_j))\epsilon D$ be any two vertices such that $\left(v_j, \sigma(v_j)\right) \notin N^S[v_i]$

Then the vertices which are not in $N_s[v_i]$ must belong to $N_s[v_j]$

Hence, any D⊂V containing this v_i and v_j will be a weight dominating set of $\overline{P_n^S}$

Thus D is a weight dominating vertex set of both P_n^S and $\overline{P_n^S}$

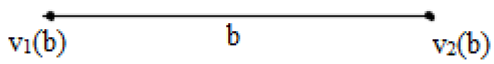
Hence D is a global dominating set of P_n^S

Since D being a γ set, it is of minimum cardinality , $\gamma^S(P_n^S)= \gamma_g^S(P_n^S)$ for n≥ 4.

Remark 4.4

For n= 2 and 3, the above theorem does not hold. This is illustrated by the following.

Consider a path on two vertices P_2^S with the semiring in Example 3.3



Here $\{v_1,v_2\}$ is a weight dominating vertex set. Also $\{v_1\}$,$\{v_2\}$ are minimal weight dominating vertex sets. Hence $\gamma^S(P_2^S) = 1$.

But $\{v_1\}$,$\{v_2\}$ are not global weight dominating vertex sets. Yet $\{v_1,v_2\}$ is a global weight dominating vertex set. Hence $\gamma_g^S(P_2^S) = 2$. Therefore $\gamma_g^S(P_2^S) \neq \gamma^S(P_2^S)$.

Consider P_3^S

Here {v_2} is a minimal weight dominating vertex set and hence $\gamma^S(P_3^S) = 1$
But $\gamma_g^S(P_3^S) = 2$

Theorem 4.5
D is a global weight dominating vertex set of P_n^S if and only if it is a global weight dominating vertex set of $(P_n^s)'$
Proof:
For P_n^S (n≥4) consider the global weight dominating vertex set as

$$D=\begin{cases}\{v_2, v_5, \dots\dots\dots v_{3j+2}\} & \text{if } n \equiv 0 \text{ or } 2(\text{mod } 3)\\ \{v_2, v_5, \dots\dots\dots\dots v_{3j+2}\} \cup \{v_{n-1}\} & if\ n \equiv 1\ mod\ (3)\end{cases} \quad \text{where } 0 \le j \le \lfloor\frac{n-2}{3}\rfloor$$

We have the following three cases when duplication of a vertex of P_n^S takes place
Case (i) and (ii):
Either a pendant vertex or an internal vertex of P_n^S not belonging to D is duplicated.
Then a duplicated vertex (v',$\sigma(v')$) is adjacent to a vertex in D and V($(P_n^S)'$) = V(P_n^S) U{ (v',$\sigma(v')$)}
Hence D is a global weight dominating vertex set of $(P_n^s)'$.
Case (iii):
An internal vertex of P_n^S of D is duplicated.
For P_n n≥ 4 , consider the global weight dominating set

$$D=\begin{cases}\{v_1, v_4, v_7 \dots\dots\dots v_{3j+1}\} & \text{if } n \equiv 1 \text{ or } 2(\text{mod } 3)\\ \{v_1, v_4, v_7 \dots\dots\dots\dots v_{3j+2}\} \cup \{v_n\} & if\ n \equiv 0\ mod\ (3)\end{cases} \quad \text{where } 0 \le j \le \lfloor\frac{n-1}{3}\rfloor$$

Here the duplicated vertex (v',$\sigma(v')$) is adjacent to a vertex in D and V($(P_n^s)'$)) = V(P_n^S) U{ (v',$\sigma(v')$) }
Hence D is a global weight dominating vertex set of $(P_n^s)'$
Converse:
Suppose that D is a global weight dominating vertex set of $(P_n^s)'$
Therefore D is a weight dominating vertex set of both $(P_n^s)'$ and $\overline{(P_n^s)'}$
But V($(P_n^s)'$) = V(P_n^S) U{ (v',$\sigma(v')$)} and V($\overline{(P_n^s)'}$) = V($\overline{P_n^S}$) U{ (v',$\sigma(v')$)}.
More over D being a global weight dominating vertex set of $(P_n^s)'$, there exists a vertex in D which will dominate both (v,σ(v)) and(v',$\sigma(v')$) in $(P_n^s)'$, as well as $\overline{(P_n^s)'}$.
Hence D is a weight dominating vertex set of both P_n^S and $\overline{P_n^S}$.
Therefore D is a global weight dominating vertex set of P_n^S
Theorem 4.6
$\gamma_g^S((P_n^s)') = \gamma_g^S(P_n^S)$, for n=2,3.
Proof:
For the path P_2^S , V(P_2^S)={ (v_1,σ(v_1)) , (v_2,σ(v_2))}
Then D={v_1,v_2} is a global dominating set of P_2^S with minimum cardinality. Hence γ_g^S (P_2^S)=2
Now on duplicating either of the pendant vertices of P_2^S by a vertex (v',$\sigma(v')$).
Then D={v_1,v_2} is a global weight dominating vertex set of $(P_2^s)'$ which is also a minimal weight dominating vertex set. Hence $\gamma_g^S((P_2^s)')$=2.
Consider the path P_3^S
V(P_3^S)={ (v_1,σ(v_1)) , (v_2,σ(v_2)), (v_3,σ(v_3))}
Then D={v_1,v_2} is a global dominating set of P_3^S which is also a minimal weight dominating vertex set.
Hence γ_g^S (P_3^S)=2

Case (i)
If (v',$\sigma(v')$) is duplicated vertex of either of the pendant vertices of P_3^S , then D={v_1,v_2} is a global weight dominating vertex set of $(P_3^s)'$ which is also a minimal weight dominating vertex set. Hence $\gamma_g^S((P_3^s)')$=2.
Case (ii)
If (v',$\sigma(v')$) is duplicated vertex of an internal vertex v of P_3^S Then D={v_1,v_2} is a global weight dominating vertex set of $(P_3^S)'$ which is also a minimal weight dominating vertex set.
Hence $\gamma_g^S((P_3^s)')$=2.
$\gamma_g^S((P_n^s)') = \gamma_g^S(P_n^S)$, for n=2,3.

V. Conclusions

In S-valued graphs, we derived some properties of global weight dominating sets in duplication of vertices related to cycles and paths. Similar results can be obtained for different types of S-valued graphs.

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